The effect of a magnetic field on Stokes flow in a conducting fluid

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SUMMARY

Low Reynolds number flow of a conducting fluid past a sphere is considered. The classical Stokes solution is modified by a magnetic field which, at infinity, is uniform and in the direction of flow of the fluid.

The formula for the drag is found to be

$$D = D_S\left\{1+\frac{3}{8}M+\frac{7}{960}M^2-\frac{43}{7680}M^3+O(M^4)\right\},$$

where D_S is the Stokes drag and M is the Hartmann number.

INTRODUCTION

We consider the streaming motion of an incompressible, viscous, conducting fluid past a sphere, in the presence of a magnetic field. At infinity the streaming motion and the magnetic field are assumed to be uniform, and their directions are parallel.

For flow in the absence of the magnetic field, the problem was first considered by Stokes (Lamb 1932) on the assumption that the Reynolds number $R = Ua/\nu$ was negligibly small, where U is the undisturbed velocity of the stream, a is the radius of the sphere, and ν the kinematic viscosity. The classical result for the drag force experienced by the sphere is, according to the Stokes approximation,

$$D_S = 6\pi\rho\nu a U,$$

where ρ is the density.

Various writers have refined this result for the case in which R is small but not negligible; in particular Oseen (Lamb 1932) found the formula

$$D = D_S(1+\tfrac{3}{8}R)$$

for the next approximation to the drag. More recently Proudman & Pearson (1957) have obtained the result

$$D = D_S\left\{1+\frac{3}{8}R+\frac{9}{40}R^2\log R+O(R^2)\right\}.$$

It is the purpose of the present paper to investigate the modification of the Stokes formula due to the presence of the magnetic field. Because of the motion of the fluid in the magnetic field, an associated electrical field

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is produced which, according to Ohm's law, sets up electrical currents in the fluid if the latter is a conductor. The interaction of these currents with the magnetic field then produces a body force which must be included in the Navier–Stokes equations for the motion of the fluid. The effect of this body force is to inhibit the motion of the fluid across the lines of force. Since, as will be shown, the lines of force are approximately in the direction of the undisturbed stream, the natural tendency of the fluid to flow round the sphere is opposed and the result is an increase in drag. The possibility of separation behind the sphere will also be enhanced, and then the present theory will no longer apply. However, there is presumably a certain range in which there is no separation, as in the Stokes problem. In this range, the theory will be valid.

The problem also seems amenable to experiment, and investigations are to be conducted at the Guggenheim Aeronautical Laboratory, California Institute of Technology, on the motion of small spheres in a column of mercury, the magnetic field being produced by a surrounding coil carrying a current.

Governing equations

The flow is assumed to be steady and parallel to the negative x-axis at infinity. The equations to be solved are then, using m.k.s. units (Cowling 1957), and with the usual notation for electromagnetic quantities,

$$\operatorname{curl}\mathbf{H} = \mathbf{j}, \qquad \operatorname{div}\mathbf{H} = 0, \tag{1}$$

$$\operatorname{curl}\mathbf{E} = 0, \tag{2}$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mu\mathbf{V}' \times \mathbf{H}), \tag{3}$$

$$\operatorname{div}\mathbf{V}' = 0, \tag{4}$$

$$\rho(\mathbf{V}' . \nabla)\mathbf{V}' = -\nabla p' + \rho\nu\nabla^2\mathbf{V}' + \mu\mathbf{j} \times \mathbf{H}, \tag{5}$$

where $\mathbf{V}'$ is the velocity of fluid and p' is the pressure.

From the first three equations we deduce that

$$\mu\sigma \operatorname{curl}\mathbf{V}' \times \mathbf{H} = \operatorname{curl}\operatorname{curl}\mathbf{H}. \tag{6}$$

Let U be the speed of the uniform stream at infinity parallel to the negative x-axis, and let a be the radius of the sphere. The space coordinates may then be made non-dimensional with the factor a^{-1} and the pressure and velocity as follows,

$$p = \frac{a}{\rho\nu U}p', \qquad \mathbf{V} = \frac{\mathbf{V}' + U\mathbf{i}}{U},$$

where $\mathbf{i}$ is a unit vector along the x-axis. Note that the boundary conditions on $\mathbf{V}$ are now $\mathbf{V} = 0$ at infinity and $\mathbf{V} = \mathbf{i}$ at the sphere.

Equations (4), (5) and (6) then become

$$\operatorname{div}\mathbf{V} = 0, \tag{7}$$

$$R(\mathbf{V} . \nabla)\mathbf{V} = -\nabla p + \nabla^2\mathbf{V} + (\mu a^2/\rho\nu U)\mathbf{j} \times \mathbf{H}, \tag{8}$$

$$R_m \operatorname{curl}\mathbf{V} \times \mathbf{H} = \operatorname{curl}\operatorname{curl}\mathbf{H}, \tag{9}$$

where
$$R = \frac{Ua}{\nu} = \text{Reynolds number},$$
$$R_m = Ua\mu\sigma = \text{magnetic Reynolds number},$$
and all the operators now refer to non-dimensional coordinates.

As in the classical Stokes problem we assume that R is small. In addition we assume that R_m is small. In most practical problems the first condition implies the second. For example, for mercury, $\nu \doteqdot 10^{-2}$ and $\mu\sigma \doteqdot 10^{-5}$, so that $R_m/R \doteqdot 10^{-7}$.

If the term containing R_m in (9) is neglected we get
$$\operatorname{curl}\operatorname{curl}\mathbf{H} = 0,$$
and so the magnetic field is independent of the fluid velocity to this approximation. In practice the permeabilities of the fluid and the sphere will be effectively equal provided both are non-magnetic, so that we may take the magnetic field to be uniform and parallel to the x-axis at all points. This makes $\mathbf{V}\times\mathbf{H} = 0$ both at the sphere and at infinity. Hence equation (2), and the boundary conditions on $\mathbf{E}$ and $\mathbf{j}$, are satisfied by
$$\mathbf{E} = 0, \qquad \mathbf{j} = U\mu\sigma\mathbf{V}\times\mathbf{H}. \tag{10}$$

The problem is now reduced to the solution of equations (7) and (8) for $\mathbf{V}$ and p, with $\mathbf{j}$ given by (10) and with a prescribed value for $\mathbf{H} = H\mathbf{i}$.

It should be noted that some of the complications which arise from boundary conditions in magneto-hydrodynamics are conveniently absent in this problem. Moreover the displacement currents are identically zero, since the motion is steady. Nor is there an electric field required to prevent a pile-up of charge on the sphere, for the current filaments, like the vortex filaments, form closed circuits coaxial with the sphere and vanish on its surface.

Solution of the equations

In equation (8) we omit the left-hand side, since it contains the factor R. The equations are then
$$\nabla\cdot\mathbf{V} = 0, \tag{11}$$
$$-\nabla p + \nabla^2\mathbf{V} - M^2\{\mathbf{V} - (\mathbf{V}\cdot\mathbf{i})\mathbf{i}\} = 0, \tag{12}$$
where M denotes the Hartmann number $\mu Ha(\sigma/\rho\nu)^{1/2}$ and is essentially non-negative.

We assume the following terms for $\mathbf{V}$ and p:
$$\mathbf{V} = e^{Mx}\nabla\phi_1 + e^{-Mx}\nabla\phi_2, \tag{13}$$
$$p = M\left(e^{Mx}\frac{\partial\phi_1}{\partial x} - e^{-Mx}\frac{\partial\phi_2}{\partial x}\right). \tag{14}$$
where ϕ_1 and ϕ_2 are to be determined. It will appear (see equations (19) and (18) below) that both ϕ_1 and ϕ_2 are $O(M^{-1})$ as $M\to 0$, so that the limiting value for the pressure (the classical Stokes solution) is non-zero.

Moreover, the singular terms in equation (13) cancel each other, so that a finite limiting value for the velocity is also obtained.

Equations (11) and (12) now become

$$\nabla.\mathbf{V} = e^{Mx}\left(\nabla^2\phi_1 + M\frac{\partial\phi_1}{\partial x}\right) + e^{-Mx}\left(\nabla^2\phi_2 - M\frac{\partial\phi_2}{\partial x}\right) = 0,$$

$$-\nabla p + \nabla^2\mathbf{V} - M^2\{\mathbf{V} - (\mathbf{V}.\mathbf{i})\mathbf{i}\} = e^{Mx}\nabla\left(\nabla^2\phi_1 + M\frac{\partial\phi_1}{\partial x}\right) + $$
$$+ e^{-Mx}\nabla\left(\nabla^2\phi_2 - M\frac{\partial\phi_2}{\partial x}\right) = 0,$$

and so both equations are satisfied provided that

$$\nabla^2\phi_1 + M\,\partial\phi_1/\partial x = 0, \tag{15}$$

$$\nabla^2\phi_2 - M\,\partial\phi_2/\partial x = 0. \tag{16}$$

Solutions of equations (15) and (16) may be written in the form

$$\phi_1 = (\pi/Mr)^{1/2}e^{-\frac{1}{2}Mx}\sum_{n=0}^{\infty} A_n K_{n+\frac{1}{2}}(\tfrac{1}{2}Mr)P_n(\cos\theta), \tag{17}$$

$$\phi_2 = (\pi/M\mathrm{r})^{1/2}e^{\frac{1}{2}Mx}\sum_{n=0}^{\infty} B_n K_{n+\frac{1}{2}}(\tfrac{1}{2}Mr)P_n(\cos\theta), \tag{18}$$

where (r, θ) are non-dimensional spherical polar coordinates, K_n is the modified Bessel function and P_n is the Legendre polynomial.

Considerations of symmetry also imply that $B_n = (-1)^{n+1}A_n$; for example the x-component of the velocity must be an even function of x.

It remains to obtain the coefficients A_n by imposing the boundary conditions that the x-component of $\mathbf{V}$ as given by (13) shall be unity and the component perpendicular to the axis zero at the sphere. For values of M which are not too large this can be done numerically, approximating to the resulting series by a finite number of terms. The expansion of $e^{\xi\cos\theta}P_n(\cos\theta)$ in terms of spherical harmonics is required, and will be found in Goldstein (1929). However, values of M which are $O(1)$ and smaller are reasonable in laboratory experiments. For example, the Hartmann number for a sphere of radius 0·1 cm in mercury subject to a field of 100 gauss is about 0·1. Thus a solution in powers of M is not without interest. For this purpose, solutions of (15) and (16) of a different form were used in order to avoid the algebraical manipulation of Bessel functions, Legendre polynomials and their derivatives.

Fundamental solutions of equations (15) and (16) are, respectively

$$r^{-1}e^{-\frac{1}{2}M(r+x)}, \qquad r^{-1}e^{-\frac{1}{2}M(r-x)},$$

and further linearly independent solutions can be formed, having the required symmetry about the x-axis and the correct behaviour at infinity, by successive differentiation with respect to x. Thus we may write

$$\phi_1 = \sum_{n=0}^{\infty} C_n\left(\frac{\partial}{\partial x}\right)^n (r^{-1}e^{-\frac{1}{2}M(r+x)}),$$

$$\phi_2 = \sum_{n=0}^{\infty} (-1)^{n+1}C_n\left(\frac{\partial}{\partial x}\right)^n (r^{-1}e^{-\frac{1}{2}M(r-x)}),$$

$$\mathbf{V} = e^{Mx}\nabla\left\{\sum_{n=0}^{\infty} C_n\left(\frac{\partial}{\partial x}\right)^n (r^{-1}e^{-\frac{1}{2}M(r+x)})\right\} - e^{-Mx}\nabla\left\{\sum_{n=0}^{\infty} (-1)^n C_n\left(\frac{\partial}{\partial x}\right)^n (r^{-1}e^{-\frac{1}{2}M(r-x)})\right\},$$

$$p = Me^{Mx}\sum_{n=0}^{\infty} C_n\left(\frac{\partial}{\partial x}\right)^{n+1} (r^{-1}e^{-\frac{1}{2}M(r+x)}) + Me^{-Mx}\sum_{n=0}^{\infty} (-1)^n C_n\left(\frac{\partial}{\partial x}\right)^{n+1} (r^{-1}e^{-\frac{1}{2}M(r-x)}).$$

For small values of M the exponential terms may be expanded in series. The differentiations are easily performed and the first few coefficients calculated so as to be consistent with the boundary conditions. The details are straightforward and are omitted. When the coefficients are obtained, the drag may be computed from the stress across the surface of the sphere using the formula (Lamb 1932)

$$rp_{rx} = -px + \{r(\partial/\partial r) - 1\}(\mathbf{V}\cdot\mathbf{i}) + \partial(\mathbf{r}\cdot\mathbf{V})/\partial x.$$

The final result for the drag is

$$D = D_S\left\{1 + \frac{3}{8}M + \frac{7}{960}M^2 - \frac{43}{7680}M^3 + O(M^4)\right\}.$$

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